

Symmetric Polynomials in Tropical Algebra Semirings

Sara Kališnik Verovšek*

Department of Mathematics, Brown University

Davorin Lešnik†

Department of Mathematics, University of Ljubljana

Abstract

The growth of tropical geometry has generated significant interest in the tropical semiring in the past decade. However, there are other semirings in tropical algebra that provide more information, such as the symmetrized $(\max, +)$, Izhakian’s extended and Izhakian-Rowen’s supertropical semirings. In this paper we identify in which of these upper-bound semirings we can express symmetric polynomials in terms of elementary ones. This allows us to determine the tropical algebra semirings where an analogue of the Fundamental Theorem of Symmetric Polynomials holds and to what extent.

1 Introduction

Tropical algebra is a relatively new branch of mathematics, which has gained a lot of popularity over the last two decade [11, 15, 12]. The adjective ‘tropical’ was coined by French mathematicians in honor of the Brazilian computer scientist Imre Simon [14], one of the pioneers in min-plus algebra. It builds on the older area more commonly known as max-plus algebra, which arises in semigroup theory, optimization, and computer science [1, 3].

*Electronic address: sara_kalisnik_verovsek@brown.edu; Corresponding author

†Electronic address: davorin.lesnik@fmf.uni-lj.si; This author was partially supported by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF under Award No. FA9550-14-1-0096.

The tropical semiring lies at the heart of tropical geometry. In simplest terms, tropical geometry can be thought of as algebraic geometry over the tropical semiring, a piecewise linear version of algebraic geometry, which replaces a variety by its combinatorial shadow. Although much work has been done, there is not yet a complete translation of the methods of algebraic geometry to the tropical situation. In particular, one of the main objects of study in algebraic geometry, invariant theory, has not been studied much in the tropical setting.

In [2], we initiated the translation of invariant theory by studying the tropical semiring. In this paper we build on our previous results and answer what happens in other semirings of interest to tropical algebraists, such as the symmetrized $(\max, +)$ semiring [3], the extended tropical semiring [5], the supertropical semiring [8, 5, 9]. They are all *upper-bound* semirings [7, 6]; therefore, we formulate the statements in this setting.

We study elementarity, i.e. the ability to express symmetric polynomials with elementary ones. Given $n \in \mathbb{N}$, we say that a semiring X is *n-elementary* when every symmetric polynomial p in n variables can be written as a polynomial in the elementary symmetric polynomials. The semiring X is *fully elementary* when it is n -elementary for all $n \in \mathbb{N}$.

We prove that in upper-bound semirings 2-elementarity is equivalent to the Frobenius property (Theorem 3.5). In idempotent semirings the Frobenius property is equivalent to full elementarity (Theorem 4.6 and Corollary 4.7). In addition, supertropical semirings, which are all Frobenius, are fully elementary (Theorem 5.9).

2 Preliminaries

Recall that $(X, +, 0, \cdot)$ is a *semiring* when $(X, +, 0)$ is a commutative monoid, (X, \cdot) a semigroup, the multiplication \cdot distributes over the addition $+$ and 0 is an absorbing element, i.e. $0 \cdot x = x \cdot 0 = 0$ for all $x \in X$.

A semiring is *unital* when it has the multiplicative unit 1 , i.e. such an element $1 \in X$ that $1 \cdot x = x \cdot 1 = x$ for all $x \in X$. A semiring is *commutative* when \cdot is commutative. It is *idempotent* when $+$ is idempotent, i.e. $x + x = x$ for all $x \in X$. A map between semirings is a semiring homomorphism when it preserves addition, zero and multiplication. If the semirings are unital, it is called a unital semiring homomorphism when it additionally preserves 1 . For an excellent introduction to the theory of semirings, we refer the reader to [4].

As usual, we often omit the \cdot sign in algebraic expressions, and shorten the product of n many factors x to x^n . Also, we write just X instead of $(X, +, 0, \cdot)$ (or $(X, +, 0, \cdot, 1)$)

when the operations are clear.

Given a unital semiring X , we can view any natural number¹ $n \in \mathbb{N}$ as an element of X in the usual way:

$$n = \underbrace{1 + 1 + \dots + 1}_{n\text{-times}}.$$

However, this mapping (in fact, a unital semiring homomorphism) from \mathbb{N} to X need not be injective; for example, if X is idempotent, then $1 = 2$. In fact, that is a characterization of idempotency in unital semirings: we get the converse by multiplying the equality $1 = 2$ with an arbitrary $x \in X$.

In any commutative monoid $(X, +, 0)$ we can define a binary relation, i.e. *intrinsic ordering*, by

$$a \leq b \text{ if and only if } \exists x \in X . a + x = b$$

for $a, b \in X$.

The intrinsic order is reflexive (since $a + 0 = a$) and transitive (since if $a + x = b$ and $b + y = c$, then $c = b + y = a + x + y$). Thus, it is a preorder on X . Note that 0 is a least element in this preorder (since $0 + a = a$), and $+$ is monotone, in the sense that if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$. Of course, any semiring is a commutative monoid for $+$ and thus has the intrinsic order. Note that in this case multiplication is monotone as well: distributive laws give us that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$, and then it follows from $a \leq b$ and $c \leq d$ that $ac \leq bc \leq bd$.

The point of the intrinsic order in this paper is the directedness it implies: for any $a, b \in X$ we can find an upper bound for them, namely $a + b$. More generally, in the proofs we repeatedly use the fact that a part of a sum (with many summands) is in relation \leq with the entirety of the sum.

Recall that any preorder defines an equivalence relation by $a \approx b := a \leq b \wedge b \leq a$. A preorder is antisymmetric (hence, a partial order) when \approx is equality. In general, a preorder on a set X induces a partial order on the quotient set X/\approx .

The intrinsic order on a commutative monoid (or a semiring) is not necessarily antisymmetric; for example, in a group (or a ring) all elements are equivalent. Yet, antisymmetry is crucial in our arguments, hence the following definition, which already appeared in [7, 6].

Definition 2.1 A semiring is *upper-bound* when its intrinsic order is antisymmetric (thus a partial order).

¹In this paper we consider 0 to be a natural number. It represents a unit for addition in X .

A simple example of an upper-bound semiring is the set of natural numbers \mathbb{N} , where the intrinsic order is the usual \leq . However, we will be particularly interested in idempotent semirings.

Proposition 2.2 *Let X be a semiring. Define a binary relation \ll on X by*

$$a \ll b \text{ if and only if } a + b = b.$$

Then \ll is antisymmetric and transitive, and the following statements are equivalent.

1. *X is an idempotent semiring.*
2. *The relation \ll is reflexive (thus a partial order).*
3. *The relation \ll is the same as the intrinsic order \leq .*
4. *For any $a, b \in X$ their sum $a + b$ is the unique join (the least upper bound) of a and b in the intrinsic order.*

In particular, any idempotent semiring is an upper-bound semiring.

Proof. See Example 5.3 [6]. ■

We now turn our attention to polynomials over semirings.

Definition 2.3 Let X be a unital commutative semiring.

- Let $m, n, d_{1,1}, \dots, d_{m,n}$ be natural numbers and $a_1, \dots, a_m \in X$. A *polynomial expression* is a syntactic object of the form $\sum_{k=1}^m a_k \prod_{j=1}^n x_j^{d_{k,j}}$.
- A *polynomial* is a function $X^n \rightarrow X$ that a polynomial expression represents.
- A polynomial expression is *symmetric* when for each monomial $a_k \prod_{j=1}^n x_j^{d_{k,j}}$ in it and each permutation $\sigma \in S_n$ the monomial $a_k \prod_{j=1}^n x_{\sigma(j)}^{d_{k,j}}$ also appears in it, up to a change of the order of factors.² A polynomial is *symmetric* when it can be represented by a symmetric polynomial expression.

²That is, we consider symmetry relative to commutativity of multiplication. For example, the polynomial expression xy is symmetric: the transposition of variables gives yx , which we identify with xy . Of course, it would not make sense to consider xy symmetric over a non-commutative semiring.

- For any $n \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ the *elementary symmetric polynomial* $\mathbf{el}_{n,j}: X^n \rightarrow X$ is the sum of all products of j different variables, i.e.

$$\begin{aligned}\mathbf{el}_{n,1}(x_1, \dots, x_n) &= x_1 + x_2 + \dots + x_n, \\ \mathbf{el}_{n,2}(x_1, \dots, x_n) &= x_1x_2 + x_1x_3 + x_2x_3 + \dots + x_{n-1}x_n, \\ &\vdots \\ \mathbf{el}_{n,n}(x_1, \dots, x_n) &= x_1x_2 \dots x_n.\end{aligned}$$

Note that $\mathbf{el}_{n,j}$ has $\binom{n}{j}$ terms.

The goal of this paper is to determine when it is possible to express symmetric polynomials in terms of elementary symmetric polynomials in certain upper-bound semirings. For the sake of neatly expressing this, we introduce the following definition.

Definition 2.4 Let X be a unital commutative semiring.

- Given $n \in \mathbb{N}$, X is *n-elementary* when for every symmetric polynomial p in n variables there exists a polynomial r in n variables, such that

$$p(x_1, \dots, x_n) = r(\mathbf{el}_{n,1}(x_1, \dots, x_n), \dots, \mathbf{el}_{n,n}(x_1, \dots, x_n))$$

for all $x_1, \dots, x_n \in X$.

- X is *fully elementary* when it is n -elementary for all $n \in \mathbb{N}$.

Any unital commutative semiring is 0-elementary and 1-elementary.

In the following four sections we discuss elementarity in different semirings.

3 Elementarity and Frobenius Equalities

As we shall see, elementarity in semirings is closely related to Frobenius equalities. Recall that the *Frobenius equality*³ for $n \in \mathbb{N}$ in a unital commutative semiring X states that $(x + y)^n = x^n + y^n$ for all $x, y \in X$.

Note that the Frobenius equality for 0 is a bit special: it states that $1 = 1 + 1$, so it is equivalent to the semiring being idempotent. For other Frobenius equalities we have the following definition.

³Also called ‘Freshman’s Dream’ for obvious reasons.

Definition 3.1 A unital commutative semiring X is *Frobenius* when it satisfies Frobenius equalities for all $n \in \mathbb{N}_{\geq 1}$.

Here is one source of Frobenius semirings.

Proposition 3.2 *Let X be an idempotent unital commutative semiring, in which the intrinsic order is linear, i.e. $x \leq y$ or $y \leq x$ for all $x, y \in X$. Then X is Frobenius.*

Proof. Let $n \in \mathbb{N}_{\geq 1}$ and $x, y \in X$. If $x \leq y$, then by monotonicity of multiplication $x^n \leq y^n$. By Proposition 2.2

$$(x + y)^n = y^n = x^n + y^n.$$

Similarly for $y \leq x$. ■

We already mentioned that the Frobenius equality for 0 amounts to the idempotency of the semiring, which can be expressed as $1 = 2$ (and consequently $m = n$ in X for all $m, n \in \mathbb{N}_{\geq 1}$, since we can keep adding 1 to both sides of the equation). Other Frobenius equalities also give us some equalities between natural numbers in a semiring.

Lemma 3.3 *Let X be a Frobenius semiring.*

1. *Then $2 = 4$ in X . Consequently, any $m, n \in \mathbb{N}_{\geq 2}$ are the same in X if they have equal parity.*
2. *If X is upper-bound, then $2 = 3$ in X . Consequently, any $m, n \in \mathbb{N}_{\geq 2}$ are the same in X .*

Proof.

1. We have $2 = 1^2 + 1^2 = (1 + 1)^2 = 4$. It follows inductively that $2 = 2k$ in X for all $k \in \mathbb{N}_{\geq 1}$.
2. By definition of \leq we have $2 \leq 2 + 1 = 3 \leq 3 + 1 = 4 = 2$. By antisymmetry of \leq we get $2 = 3$. It follows inductively that $2 = k$ in X for all $k \in \mathbb{N}_{\geq 2}$. ■

Remark 3.4 If a Frobenius semiring is not upper-bound, we might not have $2 = 3$. For example, take the semiring \mathbb{N} and define $a, b \in \mathbb{N}$ to be equivalent when they are equal or they are both ≥ 2 and of equal parity. The quotient $\mathbb{N}/\sim = \{[0], [1], [2], [3]\}$ inherits the semiring structure, for which it is Frobenius, and we have $[2] \neq [3]$. Of course, \mathbb{N}/\sim is then not upper-bound as $[0] \leq [1] \leq [2] \leq [3] \leq [2]$.

The Frobenius property allows the following partial characterization of elementarity.

Theorem 3.5 *Let X be a unital commutative semiring.*

1. *If X is Frobenius, it is 2-elementary.*
2. *If X is upper-bound, the converse also holds.*

Proof.

1. Let $p(x, y) = \sum_{k=1}^m a_k x^{i_k} y^{j_k}$ be a symmetric polynomial, represented by a symmetric polynomial expression. Hence, for any monomial $a_k x^{i_k} y^{j_k}$ in p , if $i_k \neq j_k$, the expression also possesses the monomial of the form $a_k x^{j_k} y^{i_k}$. Thus we can write

$$\begin{aligned} p(x, y) &= \sum_{k \in \{1, \dots, m\}, i_k = j_k} a_k (xy)^{i_k} + \sum_{k \in \{1, \dots, m\}, i_k > j_k} a_k (x^{i_k} y^{j_k} + x^{j_k} y^{i_k}) = \\ &= \sum_{k \in \{1, \dots, m\}, i_k = j_k} a_k (xy)^{i_k} + \sum_{k \in \{1, \dots, m\}, i_k > j_k} a_k (xy)^{j_k} (x^{i_k - j_k} + y^{i_k - j_k}). \end{aligned}$$

Since $i_k - j_k \geq 1$ in the last sum, Frobenius equality implies

$$x^{i_k - j_k} + y^{i_k - j_k} = (x + y)^{i_k - j_k}.$$

Therefore

$$p(x, y) = \sum_{k \in \{1, \dots, m\}, i_k = j_k} a_k \mathbf{el}_{2,2}^{i_k}(x, y) + \sum_{k \in \{1, \dots, m\}, i_k > j_k} a_k \mathbf{el}_{2,2}^{j_k}(x, y) \mathbf{el}_{2,1}^{i_k - j_k}(x, y).$$

We can simplify this to

$$p(x, y) = \sum_{k \in \{1, \dots, m\}, i_k \geq j_k} a_k \mathbf{el}_{2,2}^{j_k}(x, y) \mathbf{el}_{2,1}^{i_k - j_k}(x, y).$$

2. Take any $n \in \mathbb{N}_{\geq 1}$. The polynomial $x^n + y^n$ is symmetric, so by assumption we can write

$$x^n + y^n = \sum_{k=1}^m a_k \mathbf{el}_{2,2}^{i_k}(x, y) \mathbf{el}_{2,1}^{j_k}(x, y) = \sum_{k=1}^m a_k (xy)^{i_k} (x + y)^{j_k}.$$

This holds for all $x, y \in X$. Setting y to 0 and replacing x with $x + y$ yields $(x + y)^n = \sum_{k \in \{1, \dots, m\}, i_k=0} a_k (x + y)^{j_k}$. Since adding summands can only increase the value in the intrinsic order,

$$\begin{aligned} x^n + y^n &\leq \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k = (x + y)^n = \sum_{k \in \{1, \dots, m\}, i_k=0} a_k (x + y)^{j_k} \leq \\ &\leq \sum_{k=1}^m a_k (xy)^{i_k} (x + y)^{j_k} = x^n + y^n \end{aligned}$$

(note that we needed $n \geq 1$ for the first step in this chain). By antisymmetry of \leq we conclude that $x^n + y^n = (x + y)^n$.

■

Remark 3.6 In part 2 of Theorem 3.5 the assumption of X being upper-bound is necessary. For example, symmetric polynomials over \mathbb{R} can be written as polynomials of elementary symmetric ones, but Frobenius equalities do not hold.

4 Elementarity in Idempotent Semirings

We proved that in upper-bound semirings the Frobenius property is equivalent to 2-elementarity. We now turn our attention to idempotent semirings and prove that in this case the Frobenius property is equivalent to full elementarity. In [2] we proved full elementarity for the tropical semiring, \mathbb{R}_{\min} ; what follows is an adaptation of that proof that works for general idempotent Frobenius semirings.

Let p be a polynomial in $n \in \mathbb{N}$ variables. Define

$$\text{Sym}_n(p)(x_1, \dots, x_n) := \sum_{\sigma \in S_n} p(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Proposition 4.1 *Let X be an idempotent unital commutative semiring, $n \in \mathbb{N}$, $a \in X$ and $p, q: X^n \rightarrow X$ arbitrary polynomials.*

1. $\text{Sym}_n(p)$ is a symmetric polynomial.
2. A polynomial p is symmetric if and only if $p = \text{Sym}_n(p)$.
3. $\text{Sym}_n(p + q) = \text{Sym}_n(p) + \text{Sym}_n(q)$.
4. $\text{Sym}_n(a \cdot p) = a \cdot \text{Sym}_n(p)$.
5. $\text{el}_{n,j} = \text{Sym}_n(x_1 \dots x_j)$ for all $j \in \mathbb{N}_{\leq n}$.

Proof. We leave the proof to the reader. ■

Remark 4.2 Strictly speaking, we would have to write $\text{Sym}_n((x_1, \dots, x_n) \mapsto x_1 \dots x_j)$ instead of $\text{Sym}_n(x_1 \dots x_j)$ in the previous proposition since Sym_n takes a function (a polynomial) as an argument. However, we follow the usual abuse of notation and use the shorter version. But keep in mind that the function is defined on n variables (not just j) which is emphasized by the index n in Sym_n .

Remark 4.3 In general the *symmetrization* of a polynomial is defined as

$$\text{Sym}_n(p)(x_1, \dots, x_n) := \frac{1}{n!} \sum_{\sigma \in S_n} p(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

i.e. as the *average* over permutations, so that we have $p = \text{Sym}_n(p)$ for a symmetric p . Of course, this is only well defined when factorials are invertible (equivalently, when positive natural numbers are invertible) in the semiring. That is not a problem in an idempotent semiring though, as we have $1 = 2 = 3 = \dots$, and the definition of Sym_n reduces to just the sum over permutations.

The following lemma will be used as an inductive step, with j being the variable, for which we do the induction.

Lemma 4.4 *Let X be an idempotent Frobenius semiring and $n \in \mathbb{N}_{\geq 1}$. Then for all $j \in \mathbb{N}$ with $n \geq j \geq 1$ and $d_1, \dots, d_j \in \mathbb{N}$ with $d_1 \geq d_2 \geq \dots \geq d_j$*

$$\text{Sym}_n(x_1^{d_1} \dots x_j^{d_j}) = \text{el}_{n,j}^{d_j} \cdot \text{Sym}_n(x_1^{d_1-d_j} \dots x_{j-1}^{d_{j-1}-d_j}).$$

Proof. Clearly, the equality holds for $d_j = 0$, so assume hereafter that $d_j \geq 1$. We are trying to prove that

$$\sum_{\sigma \in S_n} x_{\sigma(1)}^{d_1} \dots x_{\sigma(j)}^{d_j} = \left(\sum_{\pi \in S_n} x_{\pi(1)} \dots x_{\pi(j)} \right)^{d_j} \cdot \sum_{\rho \in S_n} x_{\rho(1)}^{d_1-d_j} \dots x_{\rho(j-1)}^{d_{j-1}-d_j},$$

which by Frobenius property reduces to

$$\sum_{\sigma \in S_n} x_{\sigma(1)}^{d_1} \dots x_{\sigma(j)}^{d_j} = \sum_{\pi \in S_n} (x_{\pi(1)} \dots x_{\pi(j)})^{d_j} \cdot \sum_{\rho \in S_n} x_{\rho(1)}^{d_1-d_j} \dots x_{\rho(j-1)}^{d_{j-1}-d_j}.$$

An idempotent semiring is upper-bound, so it suffices to prove inequality in both directions.

For any permutation $\sigma \in S_n$, $x_{\sigma(1)}^{d_1} \dots x_{\sigma(j)}^{d_j} = (x_{\sigma(1)} \dots x_{\sigma(j)})^{d_j} \cdot x_{\sigma(1)}^{d_1-d_j} \dots x_{\sigma(j-1)}^{d_{j-1}-d_j}$, so every summand from the left-hand side also appears on the right-hand side. Thus

$$\text{Sym}_n(x_1^{d_1} \dots x_j^{d_j}) \leq \text{el}_{n,j}^{d_j} \cdot \text{Sym}_n(x_1^{d_1-d_j} \dots x_{j-1}^{d_{j-1}-d_j}).$$

Conversely, take any $\pi, \rho \in S_n$, and consider the summand

$$s := (x_{\pi(1)} \dots x_{\pi(j)})^{d_j} \cdot x_{\rho(1)}^{d_1-d_j} \dots x_{\rho(j-1)}^{d_{j-1}-d_j}$$

from the right-hand side. Some of the variables might appear in both parts of this product; denote $I := \{i \in \{1, \dots, j\} \mid \exists k \in \{1, \dots, j\} \cdot \rho(i) = \pi(k)\}$. Since we can arbitrarily permute the variables in the product $x_{\pi(1)} \dots x_{\pi(j)}$ without changing its value, we may assume without loss of generality that $\pi(i) = \rho(i)$ for all $i \in I$. Denote $J := \{1, \dots, j\} \setminus I$; then for any $i \in J$ (taking into account Frobenius)

$$x_{\pi(i)}^{d_j} \cdot x_{\rho(i)}^{d_i-d_j} \leq \sum_{k=0}^{d_i} \binom{d_i}{k} x_{\pi(i)}^k x_{\rho(i)}^{d_i-k} = (x_{\pi(i)} + x_{\rho(i)})^{d_i} = x_{\pi(i)}^{d_i} + x_{\rho(i)}^{d_i},$$

so

$$s \leq \prod_{i \in I} x_{\pi(i)}^{d_i} \cdot \prod_{i \in J} (x_{\pi(i)}^{d_i} + x_{\rho(i)}^{d_i}).$$

If we use distributivity to fully expand this product, we see that each summand we get also appears in $\sum_{\sigma \in S_n} x_{\sigma(1)}^{d_1} \dots x_{\sigma(j)}^{d_j}$. Since $+$ is supremum in an idempotent semiring, we conclude that

$$\text{Sym}_n(x_1^{d_1} \dots x_j^{d_j}) \geq \text{el}_{n,j}^{d_j} \cdot \text{Sym}_n(x_1^{d_1-d_j} \dots x_{j-1}^{d_{j-1}-d_j}).$$

■

Lemma 4.5 *Let X be an idempotent Frobenius semiring. Then the symmetrization of any pure monomial (i.e. monomial with coefficient 1) is a product of elementary symmetric polynomials.*

Proof. Since $\text{Sym}_n(x_1^{d_1} \dots x_n^{d_n}) = \text{Sym}_n(x_{\sigma(1)}^{d_1} \dots x_{\sigma(n)}^{d_n})$ for any permutation $\sigma \in S_n$, any symmetrization of a pure monomial in $n \in \mathbb{N}$ variables can be written as $\text{Sym}_n(x_1^{d_1} \dots x_n^{d_n})$ where $d_1 \geq d_2 \geq \dots \geq d_n$. Using Lemma 4.4 as the inductive step, we then see that

$$\text{Sym}_n(x_1^{d_1} \dots x_n^{d_n}) = \text{el}_{n,n}^{d_n} \cdot \text{el}_{n,n-1}^{d_{n-1}-d_n} \cdot \text{el}_{n,n-2}^{d_{n-2}-d_{n-1}} \cdot \dots \cdot \text{el}_{n,1}^{d_1-d_2}.$$

■

A semiring X is 2-cancellative⁴ when $x + x = y + y \Rightarrow x = y$ holds for all $x, y \in X$. Any idempotent semiring is 2-cancellative.

Theorem 4.6 *The following statements are equivalent for any unital commutative semiring X .*

1. X is fully elementary, upper-bound and 2-cancellative.
2. X is Frobenius and idempotent.

⁴We use this name because in unital semirings we can rewrite the condition as $2x = 2y \Rightarrow x = y$, i.e. we can cancel 2.

Proof.

- (1 \Rightarrow 2)

By Theorem 3.5 X is Frobenius. By Lemma 3.3 we have $2 = 4$ in X , that is, $2 \cdot 1 = 2 \cdot 2$. Cancel 2 to get $1 = 2$, which is idempotency.

- (2 \Rightarrow 1)

Any idempotent semiring is upper-bound and 2-cancellative. Now take any symmetric polynomial $p(x_1, \dots, x_n) = \sum_{k=1}^m a_k \prod_{j=1}^n x_j^{d_{k,j}}$. Using Proposition 4.1 we get

$$p(x_1, \dots, x_n) = \text{Sym}_n(p)(x_1, \dots, x_n) = \sum_{k=1}^m a_k \text{Sym}_n\left(\prod_{j=1}^n x_j^{d_{k,j}}\right).$$

By Lemma 4.5 each $\text{Sym}_n\left(\prod_{j=1}^n x_j^{d_{k,j}}\right)$ is a product of elementary symmetric polynomials, so p can also be expressed as a polynomial in elementary symmetric polynomials.

■

Corollary 4.7 *An idempotent unital commutative semiring is fully elementary if and only if it is Frobenius.*

Proof. The claim follows from Theorem 4.6, since any idempotent semiring is upper-bound and 2-cancellative. ■

5 Elementarity in Supertropical Semirings

Another way to phrase Theorem 4.6 is that as soon as an upper-bound semiring is 2-cancellative, full elementarity is equivalent to idempotency and Frobenius. This gives us a characterization of full elementarity, but not a fully satisfactory one since there is an important class of upper-bound semirings which in general are not 2-cancellative: supertropical semiringsextendedsemiring [8, 5, 9, 6]. In this section we prove that supertropical semirings are fully elementary.

We start by recalling the relevant definitions. For any semiring X define its *ghost* [8] as

$$\nu X := \{a \in X \mid a = a + a\}.$$

The elements in νX are called *ghost elements* of X and the elements in $X \setminus \nu X$ are *tangible*. Clearly $0 \in \nu X$ and if $a, b \in \nu X$, then $a + b \in \nu X$. Also, if $a \in \nu X$ and $x \in X$, then $x \cdot a \in \nu X$ and $a \cdot x \in \nu X$. In short, νX is a semiring ideal in X . We can now immediately conclude from the definition that νX is an idempotent semiring.

Define the map $\nu: X \rightarrow X$ by $\nu(x) := x + x$. Clearly, ν is an additive monoid homomorphism, although it is not necessarily a semiring homomorphism (take for example $X = \mathbb{N}$). Also, ν is a monotone map (with regard to the intrinsic order) and we have $x \leq \nu(x)$ for all $x \in X$.

By definition, νX is the set of fixed points of ν . Note that if X is unital (so we have natural numbers in X), we can write $\nu(x) = 2x$ and $\nu X = \{a \in X \mid a = 2a\}$.

Proposition 5.1 *Let X be a unital semiring. The following statements are equivalent.*

1. $2 = 4$ in X .
2. ν is a semiring homomorphism.
3. ν is a projection (i.e. $\nu \circ \nu = \nu$).
4. The image of ν is νX . In particular, the corestriction $\nu|_{\nu X}: \nu X \rightarrow \nu X$ exists.

If these statements hold, then νX is a unital (idempotent) semiring with the multiplicative unit $\nu(1)$, so $\nu|_{\nu X}$ is a unital semiring homomorphism.

Proof.

- (1 \Rightarrow 2)

We know that ν is an additive monoid homomorphism. It remains to check that ν preserves multiplication: $\nu(x) \cdot \nu(y) = 2x \cdot 2y = 4xy = 2xy = \nu(xy)$.

- (2 \Rightarrow 1)

$$2 = \nu(1) = \nu(1 \cdot 1) = \nu(1) \cdot \nu(1) = 2 \cdot 2 = 4.$$

- (1 \Rightarrow 3)

$$\nu(\nu(x)) = 2 \cdot 2x = 4x = 2x = \nu(x).$$

- (3 \Rightarrow 4)

Since νX is the set of fixed points of ν , we always have $\nu X \subseteq \text{im}(\nu)$. For the reverse inclusion, take any $x \in X$. Then $2 \cdot \nu(x) = \nu(\nu(x)) = \nu(x)$, so $\nu(x) \in \nu X$.

- $(4 \Rightarrow 1)$

Since $2 = \nu(1) \in \text{im}(\nu) = \nu X$, we get $2 = 2 + 2 = 4$.

Assume now that the given equivalent statements hold. Then for any $x \in X$ we have $\nu(1) \cdot \nu(x) = \nu(1 \cdot x) = \nu(x)$, and likewise $\nu(x) \cdot \nu(1) = \nu(x)$, so $\nu(1)$ is indeed the multiplicative unit in νX . ■

We recall the definition from [6].

Definition 5.2 A *supertropical semiring* is a unital commutative semiring which satisfies the following:

- the equivalent statements from Proposition 5.1,
- for all $a, b \in X$ with $\nu(a) \neq \nu(b)$ we have $a + b \in \{a, b\}$,
- for all $a, b \in X$ with $\nu(a) = \nu(b)$ we have $a + b = \nu(a) = \nu(b)$.

Proposition 5.3 *The following holds for any supertropical semiring X .*

1. νX is bipotent (i.e. for all $a, b \in \nu X$ we have $a + b \in \{a, b\}$), and therefore a linearly ordered idempotent semiring.
2. X is upper-bound. The restriction of \leq from X to νX matches the intrinsic order on νX .
3. X is Frobenius and $2 = 3$ in X .

Proof.

1. νX is bipotent by [7]. Hence for any elements $a, b \in \nu X$ one of them is their common upper bound $a + b$, so they must be comparable.
2. See [6](Proposition 5.7).
3. By [8](Proposition 3.7) and Lemma 3.3.

■

In any semiring we define the *strict order relation* $<$ in the expected way: $a < b$ means $a \leq b$ and $a \neq b$. The relation $<$ is irreflexive, asymmetric and transitive. If the intrinsic order \leq is a linear partial order, then $<$ satisfies the law of trichotomy: for any $a, b \in X$ exactly one of the statements $a < b$, $a = b$, $b < a$ holds.

We examine some properties of fibers of ν in supertropical semirings.


Lemma 5.4 *Let X be a supertropical semiring and $a \in X$.*

1. *The fiber $\nu^{-1}(a)$ is non-empty if and only if $a \in \nu X$. In this case a is the largest element (with regard to the intrinsic order \leq) in $\nu^{-1}(a)$.*
2. *The elements in $\nu^{-1}(a) \setminus \{a\}$ are incomparable.*

Proof.

1. Straightforward.
2. Suppose we have $x, y \in \nu^{-1}(a) \setminus \{a\}$ with $x \leq y$ and $x \neq y$. Then $x < y < a$. There exists such $u \in X$ that $x + u = y$, in particular $u \leq y$. It follows that $2u \leq 2y = a$. We cannot have $2u = a$, as that implies $y = x + u = a$, a contradiction. Thus $2u < a = 2x$, so $x = x + u = y$, another contradiction.

■

In summary, fibers of ν look like this:  — that is, a bunch of (possibly zero) incomparable (tangible) elements, with a ghost element on the top. The entire supertropical semiring is then a disjoint union of such fibers, with the ghost part linearly ordered.

Lemma 5.5 *Let X be a supertropical semiring and $a, b, x, y \in X$.*

1. *We have*

$$a + b = b \iff a < b \vee (a \leq b \wedge b \in \nu X).$$

In particular, if $a < b$, then $a + b = b$.

2. *If $b \cdot x$ is tangible, then*

$$a < b \iff a \cdot x < b \cdot x.$$

3. *If $b \cdot y$ is tangible, $a < b$ and $x \leq y$, then $a \cdot x < b \cdot y$.*

4. If y is tangible, we have

$$x < y \iff 2x < y \iff 2x < 2y.$$

5. If x and y are tangible, exactly one of the following holds: $x < y$, $y < x$ or x and y are in the same fiber of ν .

Proof.

1. Suppose $b = a + b \geq a$. If $a \neq b$, then $a < b$. If $a = b$, we have $b = a + b = 2b$.

Conversely, suppose $a \leq b$ and $b = 2b$. Then $b \leq a + b \leq 2b = b$.

Finally, suppose $a < b$. If $2a \neq 2b$, then $a + b = b$ follows from the definition of a supertropical semiring. Suppose $2a = 2b$ (so a, b are in the same fiber of ν) while still $a < b$; then necessarily $b = \nu(b)$, and consequently $a + b = \nu(b) = b$.

2. If $b \cdot x$ is tangible, then b (and x) must also be tangible, as νX is an ideal. If $a + b = b$, then $a \cdot x + b \cdot x = (a + b) \cdot x = b \cdot x$. Using the previous item of the lemma, the result quickly follows.

3. Follows easily from the previous item.

4. Suppose $x < y$. By the first item $x + y = y$, so $y = x + y = x + x + y$. Again by the first item we conclude $2x < y$.

Clearly $2x < y$ implies the other two inequalities.

Assume $2x < 2y$. By the first item $2y = 2x + 2y = 2(x + y)$, so y and $x + y$ are in the same fiber. Since $y \leq x + y$, we have only two options. The first option is that $y = x + y$, in which case $x < y$, and we are done. The second option is that $y < x + y$ and $x + y = 2y$. This implies that x and y are in the same fiber of ν , i.e. $2x = 2y$, a contradiction.

5. The ghost ideal νX is linearly ordered, so we have $2x < 2y$, $2y < 2x$ or $2x = 2y$. The claim now follows from the previous item.

■

Lemma 5.6 *Let X be a supertropical semiring.*

1. *X is a bounded join-semilattice⁵, with $\sup \emptyset = 0$ and for any $x, y \in X$*

$$\sup\{x, y\} = \begin{cases} x + y & \text{if } x \neq y, \\ x & \text{if } x = y. \end{cases}$$

2. *Take any $n \in \mathbb{N}$ and $a_1, \dots, a_n \in X$. Let $s := a_1 + \dots + a_n$ and $M := \sup\{a_1, \dots, a_n\}$. Then*

$$\begin{aligned} s \notin \nu X &\iff M \notin \nu X \wedge \exists! i \in \{1, \dots, n\}. (a_i = M) \\ &\iff M \notin \nu X \wedge \exists! i \in \{1, \dots, n\}. (a_i = M = s). \end{aligned}$$

Proof.

1. If $x = y$, then $\sup\{x, y\} = x = y$. If $2x \neq 2y$, then the upper bound $x + y$ is necessarily the least since it equals x or y . Assume now $2x = 2y =: a$ and $x \neq y$. Suppose $x = 2x$; then $x + y = 2x = x$, so again we are done. Likewise for $y = 2y$. The only remaining case is $x, y \in \nu^{-1}(a) \setminus \{a\}$, in which case it follows from Lemma 5.4 that $x + y = a = \sup\{x, y\}$.
2. If there is exactly one maximal summand a_i , then $s = M$, and the equivalence holds.

Conversely, suppose $s \notin \nu X$. We always have $M \leq s \leq 2s = 2M$, so $M, s \in \nu^{-1}(2M) \setminus \{2M\}$. By Lemma 5.4 we get $s = M$, so also $M \notin \nu X$.

The intrinsic order of νX is the restriction of the one from X , so the supremum and ν commute, i.e. $2M = \sup\{2a_1, \dots, 2a_n\}$. Since νX is linearly ordered, this supremum is attained. Let $I := \{i \in \{1, \dots, n\} \mid 2a_i = 2M\}$; then $2s = 2M = \sup\{2a_i \mid i \in I\}$.

If M differs from all a_i , then necessarily I has at least two elements, but then $s \in \nu X$ which we know is not the case. So I has exactly one index i . Then a_i is larger than the other summands, so $a_i = M$.

■

⁵A *bounded join-semilattice* is a partial order, in which we have joins (= suprema = least upper bounds) of all finite subsets. Equivalently, we need to have the join of the empty set (which is the smallest element) and of any pair of elements.

As mentioned, the scope of Theorem 4.6 is limited when it comes to supertropical semirings since 2 is in general not cancellable. In fact, it is cancellable if and only if the supertropical semiring is idempotent, as we get $2 \cdot 1 = 2 = 4 = 2 \cdot 2$.

So what can we say about elementarity in supertropical semirings? The reasoning from the previous section does not work directly, as the symmetrization Sym_n no longer has all symmetric polynomials as fixed points; for example

$$\text{Sym}_2(xy) = xy + yx = 2xy \neq xy.$$

Nor can we redefine the symmetrization as the average (rather than the sum) over permutations because natural numbers from 2 onward (which are actually all equal to 2) are not cancellable, much less invertible.

As a way of getting around that, we introduce the *minimal symmetrization* of a pure monomial in the following way. Pick such $n, j, i_1, \dots, i_j, d_1, \dots, d_j \in \mathbb{N}_{\geq 1}$ that $i_1 < i_2 < \dots < i_j \leq n$ and that d_k s are pairwise unequal. Then

$$\text{MinSym}_n((x_1 \dots x_{i_1})^{d_1} (x_{i_1+1} \dots x_{i_2})^{d_2} \dots (x_{i_{j-1}+1} \dots x_{i_j})^{d_j})$$

is defined as the sum of terms $(x_{\sigma(1)} \dots x_{\sigma(i_1)})^{d_1} \dots (x_{\sigma(i_{j-1}+1)} \dots x_{\sigma(i_j)})^{d_j}$ over all those permutations $\sigma \in S_n$ which do not put all the variable to the same power as in some already added summand. By the standard formula for permutations of multisets this gives us $\frac{n!}{i_1!(i_2-i_1)!(i_3-i_2)! \dots (i_j-i_{j-1})!(n-i_j)!}$ terms.

Observe that the elementary symmetric polynomials are special cases of minimal symmetrizations, namely $\text{el}_{n,k}(x_1, \dots, x_n) = \text{MinSym}_n(x_1 \dots x_k)$.

Lemma 5.7 *Let X be a supertropical semiring. Given $n, j, i_1, \dots, i_j, d_1, \dots, d_j \in \mathbb{N}_{\geq 1}$ with $i_1 < i_2 < \dots < i_j \leq n$ and $d_1 > d_2 > \dots > d_j$, we have*

$$\begin{aligned} & \text{MinSym}_n((x_1 \dots x_{i_1})^{d_1} \dots (x_{i_{j-1}+1} \dots x_{i_j})^{d_j}) = \\ & = \text{el}_{n, i_j}^{d_j}(x_1, \dots, x_n) \cdot \text{MinSym}_n((x_1 \dots x_{i_1})^{d_1-d_j} \dots (x_{i_{j-2}+1} \dots x_{i_{j-1}})^{d_{j-1}-d_j}) \end{aligned}$$

for all $x_1, \dots, x_j \in X$.

Proof. Clearly the statement holds if $d_j = 0$. Assume that $d_j > 0$.

Let

$$\begin{aligned} p(x_1, \dots, x_n) &:= \text{MinSym}_n((x_1 \dots x_{i_1})^{d_1} \dots (x_{i_{j-1}+1} \dots x_{i_j})^{d_j}) \quad \text{and} \\ q(x_1, \dots, x_n) &:= \text{MinSym}_n((x_1 \dots x_{i_1})^{d_1-d_j} \dots (x_{i_{j-2}+1} \dots x_{i_{j-1}})^{d_{j-1}-d_j}). \end{aligned}$$

By Lemma 4.4 the statement is true for idempotent Frobenius semirings, in particular, it holds for νX . Thus for any $x_1, \dots, x_n \in X$

$$p(\nu(x_1), \dots, \nu(x_n)) = \mathbf{el}_{n, i_j}^{d_j}(\nu(x_1), \dots, \nu(x_n)) \cdot q(\nu(x_1), \dots, \nu(x_n)).$$

Since ν is a semiring homomorphism,

$$\nu(p(x_1, \dots, x_n)) = \nu(\mathbf{el}_{n, i_j}^{d_j}(x_1, \dots, x_n) \cdot q(x_1, \dots, x_n)).$$

That is, the two sides, the equality of which we want to prove, are at the very least in the same fiber of ν .

As each summand of $p(x_1, \dots, x_n)$ is also a summand of $\mathbf{el}_{n, i_j}^{d_j}(x_1, \dots, x_n) \cdot q(x_1, \dots, x_n)$, we have $p(x_1, \dots, x_n) \leq \mathbf{el}_{n, i_j}^{d_j}(x_1, \dots, x_n) \cdot q(x_1, \dots, x_n)$.

Suppose $p(x_1, \dots, x_n) \in \nu X$; then it is the largest element in its ν -fiber (Lemma 5.4), so we have the equality we want.

From here on suppose that $p(x_1, \dots, x_n) \in X \setminus \nu X$. According to Lemma 5.6, p contains exactly one monomial m which is strictly bigger than all the others at (x_1, \dots, x_n) , and p is equal to it. Since $m(x_1, \dots, x_n) \in X \setminus \nu X$ and νX is an ideal, all variables that appear in $m(x_1, \dots, x_n)$ must be in $X \setminus \nu X$ as well.

Let $\sigma \in S_n$ be such a permutation that $m = (x_{\sigma(1)} \dots x_{\sigma(i_1)})^{d_1} \dots (x_{\sigma(i_{j-1}+1)} \dots x_{\sigma(i_j)})^{d_j}$. We claim that values of variables strictly decrease as we move from one block in m to the next. More precisely, if $t \in \{1, \dots, j-1\}$ and $u, v \in \mathbb{N}$ are such that $1 \leq u \leq i_t < v \leq n$, then $x_{\sigma(u)} > x_{\sigma(v)}$. We prove this by eliminating all other options in part 5 of Lemma 5.5.

Let m' be the monomial in p which differs from m only in having $x_{\sigma(u)}$ and $x_{\sigma(v)}$ switched. Since $m > m'$, $m > 2m'$ and $m + m' = m$ in (x_1, \dots, x_n) by Lemma 5.5. Factor out the common part of m and m' to get

$$m(x_1, \dots, x_n) + m'(x_1, \dots, x_n) = r(x_1, \dots, x_n) \cdot (x_{\sigma(u)}^d + x_{\sigma(v)}^d)$$

for a suitable monomial r and $d \in \mathbb{N}_{\geq 1}$.

If $x_{\sigma(u)}$ and $x_{\sigma(v)}$ were in the same fiber of ν , the same would hold for $x_{\sigma(u)}^d$ and $x_{\sigma(v)}^d$. Their sum would then be in νX , implying that $m(x_1, \dots, x_n) + m'(x_1, \dots, x_n)$ is in νX as well, a contradiction.

If $x_{\sigma(u)} \leq x_{\sigma(v)}$, then $m = m + m' \leq 2m' < m$ in (x_1, \dots, x_n) , likewise a contradiction.

We conclude that $x_{\sigma(u)} > x_{\sigma(v)}$.

Take any $\pi, \rho \in S_n$, and consider the summand $s := (x_{\pi(1)} \dots x_{\pi(j)})^{d_j} \cdot x_{\rho(1)}^{d_1-d_j} \dots x_{\rho(j-1)}^{d_{j-1}-d_j}$ from the right-hand side. We follow the proof of Lemma 4.4. Let

$$I := \{i \in \{1, \dots, j\} \mid \exists k \in \{1, \dots, j\} \cdot \rho(i) = \pi(k)\}.$$

Since we can arbitrarily permute the variables in the product $x_{\pi(1)} \dots x_{\pi(j)}$ without changing its value, we may assume without loss of generality that $\pi(i) = \rho(i)$ for all $i \in I$. Let $J := \{1, \dots, j\} \setminus I$; then Frobenius property implies that

$$x_{\pi(i)}^{d_j} \cdot x_{\rho(i)}^{d_i - d_j} \leq \sum_{k=0}^{d_i} \binom{d_i}{k} x_{\pi(i)}^k x_{\rho(i)}^{d_i - k} = (x_{\pi(i)} + x_{\rho(i)})^{d_i} = x_{\pi(i)}^{d_i} + x_{\rho(i)}^{d_i},$$

for any $i \in J$. It follows that

$$s \leq \prod_{i \in I} x_{\pi(i)}^{d_i} \cdot \prod_{i \in J} (x_{\pi(i)}^{d_i} + x_{\rho(i)}^{d_i}).$$

If we use distributivity to fully expand this product, we see that each summand that we get also appears in $p(x_1, \dots, x_n)$. If we get any monomial other than m , it does not change the sum (by Lemma 5.5), since it is strictly smaller than m .

We can also get m , but only in one way; once we prove this, the proof of the lemma is done. The image of $\{1, \dots, j\}$ under π and ρ has to be the same as under σ ; in particular, $I = \{1, \dots, j\}$, and $s = m$. Clearly, there is only one way to choose the appropriate term from $\mathbf{el}_{n, i_j}^{d_j}(x_1, \dots, x_n) = \text{MinSym}_n(x_1 \dots x_k)^{d_j} = \text{MinSym}_n(x_1^{d_j} \dots x_k^{d_j})$. As for the terms in $q(x_1, \dots, x_n) = \text{MinSym}_n((x_1 \dots x_{i_1})^{d_1 - d_j} \dots (x_{i_{j-2}+1} \dots x_{i_{j-1}})^{d_{j-1} - d_j})$, only one is given by ρ which is the same as σ , up to permuting variables within blocks. The other terms are strictly smaller than that (recall from Lemma 5.5 that multiplying with an element preserves $<$, as long as the result is tangible) since variables with smaller values (as shown above) appear in powers with larger exponents, and vice versa. Adding strictly smaller terms to m does not change m . ■

Lemma 5.8 *Let X be a supertropical semiring. Any minimal symmetrization of a pure monomial over X is a product of elementary symmetric polynomials.*

Proof. Follows by induction from Lemma 5.7. ■

Theorem 5.9 *Any supertropical semiring is fully elementary.*

Proof. By definition we can write any symmetric polynomial as a linear combination of minimal symmetrizations of pure monomials. The claim then follows from Lemma 5.8. ■

6 Elementarity in Symmetrized Semirings

There exists a construction for semirings which ‘symmetrizes’ them in a particular way [13, 10]. In this section we consider elementarity of such symmetrized semirings.

Given a semiring X , its *quasisymmetrization* is defined as $\mathcal{S}(X) := X \times X$ with operations

$$\begin{aligned}(a', a'') + (b', b'') &:= (a' + b', a'' + b''), \\ (a', a'') \cdot (b', b'') &:= (a' \cdot b' + a'' \cdot b'', a' \cdot b'' + a'' \cdot b').\end{aligned}$$

These make $\mathcal{S}(X)$ into a semiring which is commutative/unital/upper-bound if X is. The additive unit is $(0, 0)$, the multiplicative is $(1, 0)$. Note that X embeds into $\mathcal{S}(X)$, in the sense that $x \mapsto (x, 0)$ is an injective (unital) semiring homomorphism.

Quasisymmetrizations are not particularly interesting when it comes to elementarity. In fact, we have the following proposition.

Proposition 6.1 *Let X be an upper-bound unital commutative semiring. The following statements are equivalent.*

1. $\mathcal{S}(X)$ is fully elementary.
2. $\mathcal{S}(X)$ is Frobenius.
3. X is trivial (i.e. $X = \{0\}$).

Proof.

- $(1 \Rightarrow 2)$

By Theorem 3.5.

- $(2 \Rightarrow 3)$

We have $(2, 0) = (1, 0)^2 + (0, 1)^2 = ((1, 0) + (0, 1))^2 = (1, 1)^2 = (2, 2)$. Hence $0 = 2$, and since $0 \leq 1 \leq 2$ and X is upper-bound, we get $0 = 1$.

- $(3 \Rightarrow 1)$

Trivial.

■

However, a quasisymmetrization is generally just the first step towards constructing a new semiring. If all elements of X are additively cancellable, the relation \sim , given by $(a', a'') \sim (b', b'') := a' + b'' = a'' + b'$, is a congruence on $\mathcal{S}(X)$, i.e. an equivalence relation which respects the semiring operations. Thus the quotient $\mathcal{S}(X)/\sim$ is a well-defined semiring. In fact, it is a ring, into which X embeds via $x \mapsto [(x, 0)]$, and is the smallest one such in the suitable sense.

This is a standard construction how to turn a semiring into a ring, but in case X is not additively cancellable, a slight adjustment is required. The given \sim is not transitive, and needs to be redefined to $(a', a'') \sim (b', b'') := \exists x \in X. a' + b'' + x = a'' + b' + x$. In that case the quotient $\mathcal{S}(X)/\sim$ is again a ring and we again have the canonical homomorphism $x \mapsto [(x, 0)]$, but this is no longer an embedding (it is not injective).

To deal with this flaw, a different relation is considered in the context of tropical-like semirings (which are not additively cancellable) [3]. Recall that then \sim is given by

$$(a', a'') \sim (b', b'') \quad := \quad (a', a'') = (b', b'') \vee (a' \neq a'' \wedge b' \neq b'' \wedge a' + b'' = a'' + b')$$

for $(a', a''), (b', b'') \in \mathcal{S}(X)$.

This relation is not automatically a congruence, though; hence the following definition.

Definition 6.2 A semiring X is *symmetrizable* when \sim is a congruence, in which case $\tilde{\mathcal{S}}(X) := \mathcal{S}(X)/\sim$ is a well-defined semiring. We call $\tilde{\mathcal{S}}(X)$ the *symmetrization* of X .

It is easy to check that if X is symmetrizable, then $x \mapsto [(x, 0)]$ is an injective (unital) semiring homomorphism from X to $\tilde{\mathcal{S}}(X)$.

In this paper we limit ourselves to elementarity of upper-bound semirings, which also limits the consideration of symmetrizable semirings.

Lemma 6.3 *Let X be a unital commutative semiring. The following statements are equivalent.*

1. X is symmetrizable and $\tilde{\mathcal{S}}(X)$ is upper-bound.
2. X is idempotent, linearly ordered by its intrinsic order, and satisfies the following property: for all $a, b, x \in X$, if $a < b$, then $a \cdot x = 0$ or $a \cdot x < b \cdot x$.

Proof.

• (1 \Rightarrow 2)

We have $[(1, 0)] + [(0, 1)] = [(1, 1)]$, so $[(1, 0)] \leq [(1, 1)]$. Suppose $1 \neq 2$ in X ; then $[(1, 1)] + [(1, 0)] = [(2, 1)] = [(1, 0)]$, so $[(1, 1)] \leq [(1, 0)]$. But $[(1, 0)] \neq [(1, 1)]$, which contradicts the assumption that X is upper-bound. Thus $1 = 2$, i.e. X is idempotent.

Now let $a, b \in X$ and suppose that neither $a \leq b$ nor $b \leq a$, meaning that $a \neq a + b \neq b$, and of course also $a \neq b$. Then $(a, b) \sim (a + b, b) \sim (a + b, a) \sim (b, a)$, so by transitivity $(a, b) \sim (b, a)$. This means that $(a, b) = (b, a)$ or $a = a + a = b + b = b$, a contradiction either way. It follows that X is linearly ordered.

Finally, let $a, b, x \in X$ with $a < b$. Then $(0, b) \sim (a, b)$, hence

$$(0, b) \cdot (x, 0) \sim (a, b) \cdot (x, 0),$$

i.e. $(0, bx) \sim (ax, bx)$. If $ax = 0$, we are done. Otherwise, $ax \neq bx$. Since $a \leq b$, $ax \leq bx$ and therefore $ax < bx$.

• (2 \Rightarrow 1)

It is clear from the definition that \sim is reflexive and symmetric. To see that it is transitive, take any

$$(a', a''), (b', b''), (c', c'') \in \mathcal{S}(X)$$

with $(a', a'') \sim (b', b'') \sim (c', c'')$. If any two of these pairs are equal, we are done. Otherwise we have $a' \neq a''$, $b' \neq b''$, $c' \neq c''$, $a' + b'' = a'' + b'$ and $b' + c'' = b'' + c'$. Assume $b' < b''$ (the case $b'' < b'$ is analogous). Since $b'' \leq b'' + a' = a'' + b'$, it follows $a'' > b'$, and then likewise $b'' > a'$, so $a'' = b''$. In the same way we get $b'' = c'' > b' < b'' > c'$. Hence $a' + c'' = a' + c'' + b' = a' + b'' + c' = a'' + b' + c' = a'' + c'$, which concludes the proof of transitivity.

We have seen that \sim is an equivalence relation. To conclude that it is a congruence, we still need to see that it respects addition and multiplication.

Let $(a', a''), (b', b''), (c', c'') \in \mathcal{S}(X)$ with $(a', a'') \sim (b', b'')$. If $(a', a'') = (b', b'')$, then $(a', a'') + (c', c'') \sim (b', b'') + (c', c'')$ and $(a', a'') \cdot (c', c'') \sim (b', b'') \cdot (c', c'')$. Suppose that $a' \neq a''$, $b' \neq b''$ and $a' + b'' = a'' + b'$. If $a' < a''$, then $a' < a'' + b' = b''$, so $b' \leq b''$, but $b' \neq b''$, so $b' < b''$. The case $a'' < a'$ is similar. Thus our assumption reduces to

$$a' < a'' \wedge b' < b'' \wedge a'' = b'' \quad \text{or} \quad a'' < a' \wedge b'' < b' \wedge a' = b'.$$

The two cases are analogous, so without loss of generality we restrict ourselves to the first one.

Suppose that $(a', a'') + (c', c'') = (b', b'') + (c', c'')$; then

$$(a', a'') + (c', c'') \sim (b', b'') + (c', c'').$$

It cannot happen that $a'' + c'' \neq b'' + c''$ since $a'' = b''$. The only remaining case is that $a' + c' \neq b' + c'$. The pairs (a', a'') and (b', b'') appear symmetrically throughout, so assume without loss of generality that $a' + c' < b' + c'$. From here it follows that $a' < b'$ and $c' < b'$. To summarize: $a', c' < b' < b'' = a''$.

Hence $a' + c' < b' + c' = b' < b'' \leq b'' + c'' = a'' + c''$, meaning that $a' + c' \neq a'' + c''$ and $b' + c' \neq b'' + c''$. Additionally, $a' + c' + b'' + c'' = a'' + c'' + b' + c'$ since $a' + b'' = a'' + b'$. We conclude $(a', a'') + (c', c'') \sim (b', b'') + (c', c'')$.

As for products, we separate three cases.

- If $(a', a'') \cdot (c', c'') = (b', b'') \cdot (c', c'')$, then $(a', a'') \cdot (c', c'') \sim (b', b'') \cdot (c', c'')$.
- Suppose $a'c' + a''c'' \neq b'c' + b''c''$. Since (a', a'') and (b', b'') appear symmetrically, we assume without loss of generality that $a'c' + a''c'' < b'c' + b''c''$; since $a'' = b''$, it follows that $a'c' < b'c'$ and $a' < b'$. Also,

$$b''c'' = a''c'' \leq a'c' + a''c'',$$

so $b'c' > b''c''$, and therefore $c' > c''$ (since $b' < b''$). Moreover,

$$b'c' = b'c' + b''c'' = a'c' + a''c'' = a'c' + b''c'',$$

so $a'c' > a''c'' = b''c''$.

Since $b' < b''$ and $b'c' \neq 0$ (because $b'c' > b''c'' \geq 0$), we have $b'c' < b''c'$. Thus $b'c'' + b''c' = b''c' > b'c' = b'c' + b''c'' > a'c' + a''c''$ and $a'c'' + a''c' = a''c' = b''c'$, so $a'c' + a''c'' \neq a'c'' + a''c'$ and $b'c' + b''c'' \neq b'c'' + b''c'$.

Also,

$$\begin{aligned} a'c' + a''c'' + b'c''b''c' &= (a' + b'')c' + (a'' + b')c'' = \\ &= (a'' + b')c' + (a' + b'')c'' = a''c' + a'c'' + b'c' + b''c''. \end{aligned}$$

In conclusion, $(a', a'') \cdot (c', c'') \sim (b', b'') \cdot (c', c'')$.

- The remaining case $a'c'' + a''c' \neq b'c'' + b''c'$ works the same as the previous one, just with c' and c'' switched.

We have seen that for any $a, b, c \in \mathcal{S}(X)$, if $a \sim b$, then $a + c \sim b + c$ and $a \cdot c \sim b \cdot c$. But then for any $a, b, c, d \in \mathcal{S}(X)$ with $a \sim b$ and $c \sim d$ we have $a + c \sim b + c \sim b + d$ and $a \cdot c \sim b \cdot c \sim b \cdot d$. We already know that \sim is transitive, so we conclude that \sim is a congruence.

■

This lemma tells us, the symmetrizations of which semirings we should consider, but the result is the same as in the case of Proposition 6.1 (with essentially the same proof).

Proposition 6.4 *Let the semiring X satisfy the equivalent properties from Lemma 6.3. The following statements are equivalent.*

1. $\tilde{\mathcal{S}}(X)$ is fully elementary.
2. $\tilde{\mathcal{S}}(X)$ is Frobenius.
3. X is trivial.

Proof.

- $(1 \Rightarrow 2)$

By Theorem 3.5.

- $(2 \Rightarrow 3)$

We have $[(1, 0)] = [(1, 0)]^2 + [(0, 1)]^2 = ([(1, 0)] + [(0, 1)])^2 = [(1, 1)]^2 = [(1, 1)]$, i.e. $(1, 0) \sim (1, 1)$. Since $(1, 1)$ is equivalent only to itself, we conclude $0 = 1$.

- $(3 \Rightarrow 1)$

Trivial.

■

7 Discussion

We studied elementarity — the ability to express symmetric polynomials with elementary ones — in upper-bound semirings. We have seen that 2-elementarity is equivalent to the Frobenius property (Theorem 3.5). We have seen that in idempotent semirings the Frobenius property is equivalent to full elementarity (Theorem 4.6 and Corollary 4.7).

Furthermore, we proved that supertropical semirings (known to be Frobenius) are all fully elementary (Theorem 5.9).

We gave a characterization for when the symmetrization of a semiring exists and yields an upper-bound semiring (Lemma 6.3). We then showed that under these conditions no non-trivial symmetrization is Frobenius or fully elementary (Proposition 6.4).

One of the goals of this paper was to answer the questions posed in [2]. The following theorem proves that the symmetrized min-plus and the symmetrized max-plus semirings are not fully elementary.

Theorem 7.1 *The tropical semiring \mathbb{R}_{\min} and the max-plus semiring \mathbb{R}_{\max} are fully elementary. Their symmetrizations $\tilde{\mathcal{S}}(\mathbb{R}_{\min})$ and $\tilde{\mathcal{S}}(\mathbb{R}_{\max})$ are not.*

Proof. \mathbb{R}_{\min} and \mathbb{R}_{\max} are unital, commutative and idempotent. The intrinsic order on \mathbb{R}_{\max} is the usual one \leq on \mathbb{R} , and on \mathbb{R}_{\min} it is its opposite \geq ; in both cases we get a linear order. Now apply Proposition 3.2 and Corollary 4.7 to get full elementarity.⁶

⁶Actually, \mathbb{R}_{\min} and \mathbb{R}_{\max} are also supertropical semirings, so we could have applied Theorem 5.9 as well.

Their symmetrizations are not fully elementary by Proposition 6.4. ■

Supertropical semirings, including the extended tropical semiring, are fully elementary.

Theorem 7.2 *The extended tropical semiring is fully elementary.*

Proof. Since the extended tropical semiring is a supertropical semiring, Theorem 5.9 applies. ■

The most general family of semirings, for which we managed to prove full elementarity, are supertropical semirings (Theorem 5.9). In other words, we have an analogue of the Fundamental Theorem of Symmetric Polynomials for supertropical semirings. Or rather, we have the existence part of this theorem. The uniqueness clearly does not hold; for example, in any Frobenius idempotent semiring the symmetric polynomial $x^2 + y^2$ can be written as $(x + y)^2$, as well as $(x + y)^2 + xy$. We do not know yet, whether a different notion of uniqueness could be defined that would allow a complete translation of the Fundamental Theorem of Symmetric Polynomials.

Our findings raise a host of further question. While we have a characterization of 2-elementarity for upper-bound semirings, our theorems about full elementarity were not as general.

Question 7.3 Does the Frobenius property characterize full elementarity in general upper-bound semirings? If not, what additional conditions are required?

Speaking of the Frobenius property, recall that it automatically holds in any linearly ordered upper-bound unital commutative idempotent semiring (Proposition 3.2), as well as in supertropical semirings which are very close to being linearly ordered. This leads to a question, in how general of semirings can we use linearity to prove Frobenius? In particular, note that (recall Lemma 3.3) we have the implications

$$1 = 2 \implies \text{Frobenius} \implies 2 = 3$$

in any linearly ordered upper-bound unital commutative semiring. The first implication does not reverse (for example, the extended tropical semiring is supertropical and thus Frobenius, but is not idempotent); what about the second one?

Question 7.4 Consider an upper-bound unital commutative semiring, linearly ordered by its intrinsic order. Is $2 = 3$ not just a necessary, but also a sufficient condition for such a semiring to be Frobenius?

Next, consider the results about semiring symmetrizations (Section 6). They were very much negative, but maybe that is because the scope was too narrow — we know

from Lemma 6.3 that if the symmetrization $\tilde{\mathcal{S}}(X)$ exists and is upper-bound, the semiring X is necessarily linearly ordered (among other things).

Question 7.5 Is there a reasonable adaptation of the symmetrizing relation \sim which also works for suitable non-linearly ordered semirings (and reduces to the usual \sim when the semiring satisfies the equivalent conditions in Lemma 6.3)?

Finally, we return to the notion of symmetric polynomials. We defined it (Definition 2.3) on the syntactic level — that is, we need to have a symmetric polynomial *expression* — since it is polynomial expressions that we work with. But one also has a reasonable notion of symmetry of polynomials on the semantic level, namely invariance under permutations of variables: $p(x_1, \dots, x_n) = p(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for all points (x_1, \dots, x_n) and permutations $\sigma \in S_n$.

Obviously the syntactic definition implies the semantic one. For idempotent semirings the converse is also true since if p is ‘semantically’ symmetric, clearly $p = \text{Sym}_n(p)$. What about more general semirings?

Question 7.6 Is it the case for every upper-bound semiring that a polynomial, invariant under permutations of variables, is necessarily symmetric? If not, is it true at least for supertropical semirings?

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